Gravity waves in a stratified fluid

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A unified treatment of wave motion in a stratified fluid, with or without density discontinuities, is achieved by reducing the governing differential system to a Sturm–Liouville system. With the aid of Sturm's comparison theorem, it is found (without detailed calculations) that, for any stratification, the phase velocity increases as the wave-number decreases and that, for the same wave-number, the phase velocity increases as the density gradient is increased everywhere and decreases as the density is increased everywhere by a constant amount. Sturm's oscillation theorem provides upper and lower bounds for the phase velocity for a given stratification, a given wave-number, and a given number of zeros of the eigenfunction (or a given number of stationary surfaces in the fluid). The inequalities giving these bounds are used to explain the well-known tendency for surfaces of density discontinuities to behave as rigid boundaries when the stratification in each layer is slight. The rigid-boundary behaviour of interfaces in such cases enables one to obtain the approximate eigenvalue spectrum by superimposing the spectra of the individual layers (with the interfaces treated as rigid) on the spectrum of the interfacial (or free surface) waves, obtained by ignoring the slight continuous stratification in each layer. It is pointed out that the Ritz method can be used for calculating the eigenvalues even when the density is discontinuous, and examples are given to show the accuracy of the Ritz method. The nature of the spectrum when the depth is infinite is also clarified.

In the course of the development of the theory, the effects of compressibility and of three-dimensionality are determined and given explicitly, the rate of growth of unstable stratifications is related to the phase velocity of waves in stable ones, and equipartition of energy is proved. Motion due to a wave-maker is discussed in order to bring out the connexion between the type of the governing partial differential equation and the nature (local or not local) of the disturbances. The effect of surface tension and the stability of a stratified fluid under vertical oscillation are also discussed.

1. Introduction

The propagation of gravity waves in a system consisting of many distinct layers of fluids was investigated by Webb (1884) and Greenhill (1887) many years ago. Recently, Benton (1953) considered the propagation of long waves in a system of flowing layers, and by a limiting process generalized the result to apply to waves propagating in a flowing fluid with continuous stratification. His approach provides a desirable link between results for distinct layers and those for a continuously stratified fluid. However, a unified theory is still lacking, and a convenient method for determining the phase velocity remains to be adopted. In this paper, the approach is diametrically opposite to that of Benton. Instead of considering a continuous stratification as a limit of a discontinuous one, as Benton did, one deals with continuous stratifications directly and treats surfaces of density discontinuity as limits of regions of large density gradients. If the effect of viscosity is neglected, the governing differential system is a Sturm–Liouville system, and Sturm's theorems can be used for the prediction of the ranges in which the phase velocities for the various modes must lie, for comparing the phase velocity in one stratified fluid with that of the corresponding mode in another, and for explaining a well-known behaviour of the surfaces of density discontinuity. The theory of infinitesimal waves presented herein also includes a technique to obtain the phase velocity to any degree of accuracy, an example of wave motion generated by a simple wave-maker, and an investigation of stability under vertical vibration.

2. The governing differential system

The differential system governing the propagation of gravity waves in a continuously stratified fluid at rest is well known (Lamb 1945, p. 378). Since the effect of compressibility will be discussed later, it is desirable to derive the differential system for wave propagation in a stratified and compressible fluid at rest, on the assumption that the change of state for each material particle is isentropic (the entire fluid not necessarily having the same entropy). The equations for incompressible fluids can then be immediately obtained by letting the sound velocity approach infinity.

With x, y, and z denoting Cartesian co-ordinates, z being measured vertically upward, and u, v and w denoting the corresponding velocity components, the mean density $\bar{\rho}(z)$ and mean pressure $\bar{p}(z)$ are related by the hydrostatic condition

$$\bar{p}' = -g\bar{\rho}. \tag{1}$$

The linearized equation of continuity is

$$\frac{\partial \rho}{\partial t}+\bar{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+w\bar{\rho}' = 0, \tag{2}$$

in which ρ is the density fluctuation and the accent indicates differentiation with respect to z. Since the velocity and the density perturbation are assumed to be small, their products and products of their derivatives have been neglected in the equation of continuity, and will be neglected in all the equations to be presented in this section. The linearized equation of isentropy is

$$\frac{\partial p}{\partial t}+w\bar{p}' = \frac{\partial p}{\partial t}-g\bar{\rho}w = c_s^2\left(\frac{\partial \rho}{\partial t}+w\bar{\rho}'\right), \tag{3}$$

in which p is the pressure fluctuation and c_s the sound velocity, which can vary with z. The equations of motion are, for an inviscid fluid,

$$\bar{\rho}\frac{\partial}{\partial t}(u,v,w) = -\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)p + (0,0-g\rho). \tag{4}$$

From the first two of equations (4), it follows immediately upon cross-differentiation that

$$\frac{\partial}{\partial t}\left(\frac{\partial(\bar{\rho}u)}{\partial y} - \frac{\partial(\bar{\rho}v)}{\partial x}\right) = 0.$$

If the motion is started from rest, or if u and v (as well as other dependent variables) are assumed to have an exponential time factor,

$$\frac{\partial(\bar{\rho}u)}{\partial y} - \frac{\partial(\bar{\rho}v)}{\partial x} = 0,$$

and a potential ϕ exists for u and v

$$\bar{\rho}u = -\frac{\partial\phi}{\partial x}, \quad \bar{\rho}v = -\frac{\partial\phi}{\partial y}.$$

The motion is therefore irrotational when *viewed from above*, in much the same way that Hele–Shaw flows are irrotational when viewed in a direction perpendicular to the (closely spaced) plane boundaries. The equation of continuity then assumes the form

$$\nabla^2\phi = \bar{\rho}\frac{\partial w}{\partial z} + w\bar{\rho}' + \frac{\partial\rho}{\partial t}, \quad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{2a}$$

and integration of the first two of equations (4) produces

$$\frac{\partial\phi}{\partial t} = p,$$

with the function of integration $F(z,t)$ absorbed in $\partial\phi/\partial t$. After differentiation with respect to t and utilization of equations (3) and (2a), this equation assumes the form

$$\frac{\partial^2\phi}{\partial t^2} = g\bar{\rho}w + c_s^2\left(\nabla^2\phi - \bar{\rho}\frac{\partial w}{\partial z}\right). \tag{5}$$

Substituting $\partial\phi/\partial t$ for p in the third of equations (4), one has

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z} + \bar{\rho}w\right) + g\rho = 0. \tag{6}$$

From equations (6) and (2a), the equation

$$\frac{\partial^3\phi}{\partial t^2\partial z} + g\nabla^2\phi - g\frac{\partial}{\partial z}(\bar{\rho}w) + \frac{\partial^2}{\partial t^2}(\bar{\rho}w) = 0$$

is obtained, whereas differentiation of equation (5) produces

$$\frac{\partial^3\phi}{\partial t^2\partial z} - g\frac{\partial}{\partial z}(\bar{\rho}w) - \frac{\partial}{\partial z}(c_s^2\nabla^2\phi) + \frac{\partial}{\partial z}\left(\bar{\rho}c_s^2\frac{\partial w}{\partial z}\right) = 0.$$

From these one obtains by subtraction

$$\nabla^2\left[g\phi+\frac{\partial}{\partial z}(c_s^2\phi)\right]+\frac{\partial^2}{\partial t^2}(\bar\rho w)-\frac{\partial}{\partial z}\left(\bar\rho c_s^2\frac{\partial w}{\partial z}\right)=0.$$

Equation (5) and the equation following (7) give

$$\frac{\partial^2}{\partial t^2}\left(g\phi+c_s^2\frac{\partial\phi}{\partial z}\right)-g^2\bar\rho w-c_s^2 g\bar\rho' w+c_s^2\frac{\partial^2}{\partial t^2}(\bar\rho w)=0,$$

from which follows

$$-(c_s^2)'\frac{\partial^2}{\partial t^2}\nabla^2\phi=\frac{\partial^4}{\partial t^4}(\bar\rho w)-\frac{\partial^3}{\partial t^2\partial z}\left(\bar\rho c_s^2\frac{\partial w}{\partial z}\right)+(g^2\bar\rho+c_s^2 g\bar\rho')\nabla^2 w-c_s^2\bar\rho\frac{\partial^2}{\partial t^2}\nabla^2 w. \quad (7)$$

In the problems treated in this paper, ϕ, w, p and ρ are assumed to have a common factor $S(x,y)$ which satisfies the equation

$$(\nabla^2+\alpha^2)S(x,y)=0. \quad (8)$$

If the time-dependence of ϕ and w is assumed to be contained in the exponential factor $e^{-i\sigma t}$, so that

$$(\phi,w)=e^{-i\sigma t}S(x,y)[\phi(z),w(z)],$$

elimination of ϕ from equations (5) and (7) with the aid of (8) produces, with w now indicating $w(z)$,

$$\left(\frac{\alpha^2c_s^2}{\sigma^2}-1\right)[\sigma^2c_s^2(\bar\rho w')'+(\sigma^4-\alpha^2g^2-c_s^2\alpha^2\sigma^2)\bar\rho w-c_s^2\alpha^2 g\bar\rho' w]$$
$$+(c_s^2)'\bar\rho(\alpha^2 gw-\sigma^2 w')=0. \quad (9)$$

For three-dimensional sinusoidal waves, the appropriate form for $S(x,y)$ is $\exp i(kx+ly)$, so that, with $\sigma=kc$,

$$(u,v,w,p,\rho)=[u(z),v(z),w(z),p(z),\rho(z)]\exp i(kx+ly-kct).$$

If, for brevity, u is written for $u(z)$, etc., the relationships between the several unknowns are

$$c\bar\rho u=p,\quad lu=kv,\quad ikc\bar\rho w=c(\bar\rho u)'+g\rho,\quad -ikcp-g\bar\rho w=c_s^2(-ikc\rho+w\bar\rho'),$$

and

$$i\rho u=\frac{kc_s^2\bar\rho}{(k^2+l^2)c_s^2-k^2c^2}\left(\frac{gw}{c_s^2}-w'\right),$$

which are recorded here for general convenience. With $\alpha^2=k^2+l^2$ and $\sigma=kc$, equation (9) has the form

$$\left[\frac{(k^2+l^2)c_s^2}{k^2c^2}-1\right]\{k^2c^2c_s^2(\bar\rho w')'+(kc)^4\bar\rho w-(k^2+l^2)[(g^2+k^2c^2c_s^2)\bar\rho w-c_s^2g\bar\rho' w]\}$$
$$+(c_s^2)'\bar\rho[(k^2+l^2)gw-k^2c^2w']=0. \quad (9a)$$

If c_s is constant, equation (9a) assumes the simpler form

$$(\bar\rho w')'-\frac{k^2+l^2}{k^2}\left(k^2\bar\rho-\frac{g}{c^2}\bar\rho'\right)w+\frac{c^2}{c_s^2}\left(k^2-\frac{(k^2+l^2)g^2}{k^2c^4}\right)\bar\rho w=0. \quad (10)$$

If the fluid is bounded by two rigid barriers at $z = 0$ and $z = d$, the boundary conditions are

$$w(0) = 0, \quad w(d) = 0.$$

Boundary conditions at free surfaces and interfaces will be presented later.

For an incompressible fluid, $c_s = \infty$, and equation (10) reduces to

$$(\bar{\rho}w')' - \frac{k^2+l^2}{k^2}\left(k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w = 0, \tag{11}$$

which for two-dimensional motion can be further reduced to the simple form

$$(\bar{\rho}w')' - \left(k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w = 0. \tag{12}$$

For two-dimensional motion of a compressible fluid with constant c_s, equation (10) has the form

$$(\bar{\rho}w')' - \left(k^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w + \frac{c^2}{c_s^2}\left(k^2 - \frac{g^2}{c^4}\right)\bar{\rho}w = 0. \tag{13}$$

3. General considerations

Although equation (10) appears rather complex as it stands, for many problems its solution can be reduced to that of equation (12). First of all, the effect of three-dimensionality can be determined once and for all. From equation (9) it is seen that for a given stratification σ^2 is a function of α^2 (in this case k^2+l^2) alone. (The sum k^2+l^2 really is the square of the wave-number of a corresponding two-dimensional wave travelling in the direction with direction numbers k, l, and zero.) Thus, without any loss of generality, the problem of determining σ given k and l is reduced to the problem of determining σ for a two-dimensional wave motion with wave-number $k_2 = \alpha$, from equation (9). The rule of conversion, first enunciated and proved by Squire (1933) in connexion with a problem of hydrodynamic stability, is as follows. The σ for a three-dimensional disturbance of wave-numbers k and l is the same as that for a two-dimensional disturbance of wave-number $(k^2+l^2)^{\frac{1}{2}}$, the actual phase velocity c (in the x-direction) is σ/k which is greater than the c for an actual two-dimensional disturbance of wave-number $(k^2+l^2)^{\frac{1}{2}}$ by the factor $(k^2+l^2)^{\frac{1}{2}}/k$.

Turning now to equation (13), one seeks to determine the effect of compressibility in a simple way, without any detailed calculations. In this connexion it must be remembered that the sound velocity is in general a function of location. For liquids it can without great error be taken to be constant, and at all events the effect of compressibility is small. For gases, c_s is constant only for an isothermal atmosphere. Under the assumption that c_s is constant, and with a fictitious wave-number k_i for a corresponding wave motion in incompressible fluid defined as

$$k_i^2 = k^2 + \frac{c^2}{c_s^2}\left(\frac{g^2}{c^4} - k^2\right), \tag{14}$$

equation (13) is reduced to the form

$$(\bar{\rho}w')' - \left(k_i^2\bar{\rho} + \frac{g}{c^2}\bar{\rho}'\right)w = 0.$$

After c is determined from this equation for a chosen k_i, the actual k can then be calculated simply from equation (14). Of course, the assumed k_i should be greater than g/cc_s (in most cases) for a *wave* motion. Otherwise k^2 would be negative (since c is in most cases less than c_s) and the motion would not be a wave motion. With k found, σ is also known. This procedure evidently involves a process of trial and error if c is to be found for a given k, or k is to be found for a given σ, but it certainly is a convenient means of determining the effect of compressibility. Since g/k is the c^2 for free-surface waves in a semi-infinite fluid, and the c^2 for internal or interfacial waves is usually smaller, $k_i^2 > k^2$ in most cases of practical interest. In a subsequent paragraph it will be shown that the c^2 determined from equation (12) decreases with k^2. Thus, whenever c_s is constant and $g > kc^2$, the effect of compressibility is to *reduce* the phase velocity, and the amount of reduction can be simply determined in the manner described above.

Attention will then be focused on equation (12) in the major part of this paper. Since this corresponds to two-dimensional flows of an incompressible fluid, the stream function will be introduced for convenience

$$\psi = f(z)\exp ik(x-ct).$$

The velocity components being

$$u = -\frac{\partial\psi}{\partial z} = -f'(z)\exp ik(x-ct), \quad w = \frac{\partial\psi}{\partial x} = ikf(z)\exp ik(x-ct),$$

the function $w(z)$ in equation (12) can be replaced by $f(z)$

$$(\bar{\rho}f')' - \left(k^2\bar{\rho} + \frac{g\bar{\rho}'}{c^2}\right)f = 0. \tag{12a}$$

If, further, the new variable

$$\eta = \frac{z}{d}$$

is introduced, and accents are now used to denote differentiations with respect to η, equation ($12a$) becomes

$$(\bar{\rho}f')' - \left(m^2\bar{\rho} + \frac{gd}{c^2}\bar{\rho}'\right)f = 0, \tag{15}$$

in which $m = kd$ is the dimensionless wave-number.

The boundary conditions at the rigid boundaries are, in terms of η,

$$f(0) = 0, \quad f(1) = 0. \tag{16}$$

At a surface of density discontinuity the density below the surface will be denoted by $\bar{\rho}_l$ and that above by $\bar{\rho}_u$. The vertical velocity w at the interface is $\partial\zeta/\partial t$, in which ζ is the deviation of the interface from its mean position. With ζ expressed as $\zeta_0 \exp ik(x-ct)$, the kinematic condition is

$$-c\zeta_0 = f, \tag{17}$$

to be applied at the interface. Apart from the exponential factor, the pressure at the interface is then, from equations (1), (17), and the first of equations (4),

$$\bar{\rho}_u\left(-cf' + \frac{g}{c}f\right)_u$$

for the upper fluid, and

$$\bar{\rho}_l\left(-cf'+\frac{g}{c}f\right)_l$$

for the lower. Continuity of the vertical velocity demands the continuity of the stream function:

$$f_u = f_l.$$

Hence the equality of pressures across the interface demands that

$$(\bar{\rho}f')_u-(\bar{\rho}f')_l+(\bar{\rho}_l-\bar{\rho}_u)\frac{gd}{c^2}f = 0, \tag{18}$$

in terms of η. Equation (18) has been obtained by a fairly complicated argument. But if $\bar{\rho}$ and f' in equation (15) are allowed to be discontinuous, and only the continuity of f is maintained, integration of that equation in the Stieltjes sense across the interface produces (18) immediately. There are then two possible approaches. One can consider the system of differential equations governing the motion in the various continuously stratified layers, together with the interfacial conditions derived above. Or one can consider the motion of the entire fluid to be governed by equation (15), and allow solutions with discontinuous f' at the interfaces. The former is the conventional approach, but the latter approach is adopted in this paper, and it appears to be the more powerful and fruitful.

By the use of the present approach the whole power of the Sturm–Liouville theory can be borrowed to achieve a unified treatment of wave motion in stratified fluids. Without detailed calculations, certain conclusions can be drawn in regard to the variation of the phase velocity with wave-length and with the density distribution, the range in which the phase velocity must lie can be predicted, and certain well-known effects of density discontinuities can be explained. Furthermore, the Ritz method can be applied for calculating the phase velocities even when the density has discontinuities. The unified treatment is the main contribution of this paper. Many of the results obtained as a consequence of this treatment are new, or have a greater generality than has previously been achieved.

In order to have a Sturm–Liouville system, the upper boundary is assumed to be always rigid, and a free surface is considered to be a liquid-gas interface covered by a fluid layer of small but non-zero* density which is bounded above by a rigid plane. The location of the upper boundary is, in the presence of a free surface, assumed to have little effect on the phase velocities. Thus the boundary condition at the upper boundary is always $f(1) = 0$, and the governing differential system is a Sturm–Liouville system consisting of equations (15) and (16) in the former of which $\bar{\rho}$ may have finite discontinuities. The integral form of equation (15) is

$$f(\eta) = \int_0^\eta \frac{1}{\bar{\rho}}\left[\bar{\rho}(0)f'(0)+\int_0^\eta (m^2\bar{\rho}+\lambda\bar{\rho}')f\,d\eta\right]d\eta, \tag{19}$$

* Dr F. Ursell has pointed out to the author that the assumption of non-zero density for the top layer, made here for convenience, is really not necessary, and the developments presented here can well be extended to include cases in which a free surface in the ordinary sense is present. In this connexion, see Sz.-Nagy (1947).

in which $\lambda = gd/c^2$, and $\bar{\rho}$ is assumed to be greater than zero. In the presence of density discontinuities, the integration in equation (20) is in the Stieltjes sense, and the function $f(\eta)$ obtained (by step-wise integration, for instance) is continuous with discontinuous derivatives at the locations where the density is discontinuous. The boundary condition $f(0) = 0$ is automatically satisfied. Since $\bar{\rho}'$ is uniformly negative, it is easy to see that a sufficiently large λ will force f to be zero at $\eta = 1$, and that at least a first eigenvalue exists. The eigenfunction satisfies equation (15) everywhere except at the density discontinuities, where equation (18) is automatically satisfied.

To show that equations (19) and (16) possess infinitely many eigenvalues corresponding to eigenfunctions with the number of zeros increasing with the index of the eigenvalues, even when $\bar{\rho}$ is discontinuous, one may approximate the given stratification by an infinite sequence of continuous stratifications with increasingly greater density gradients near the discontinuities of the given stratification. For a specified number of zeros for f in the closed interval (0 to 1 inclusive), the eigenvalues (of λ) for the sequence must approach a limit* (by the Weierstrass–Bolzano theorem), which is the eigenvalue for the given stratification, for the specified number of zeros of f in the interval. That the limit is unique follows essentially from the fact that, for a specified number of zeros of f, the eigenvalue varies continuously with a continuous variation of the density distribution—a fact that can be proved easily. Consequently much of the Sturm–Liouville theory can be carried over to the case of discontinuous density. This fortunate situation is entirely due to the fact that the interfacial conditions (18) are implied in equation (15), and automatically satisfied by equation (19).

For a given density distribution and a given wave-number, the admissible values of c are determined by equations (15) and (16). These values are the phase velocities of waves propagating in the fluid (otherwise at rest). If c has an imaginary part, the waves will grow† in amplitude. It is easy to show that if $\bar{\rho}'$ is everywhere negative c is real, and if $\bar{\rho}'$ is everywhere positive c is purely imaginary. Thus, if equation (15) is multiplied by the complex conjugate of f and integrated between 0 and 1, and equations (16) are utilized, we have

$$-\int_0^1 \bar{\rho}\,|f'|^2\,d\eta - m^2\int_0^1 \bar{\rho}\,|f|^2\,d\eta - \frac{gd}{c^2}\int_0^1 \bar{\rho}'\,|f|^2\,d\eta = 0, \tag{20}$$

which states that if $\bar{\rho}'$ is negative throughout, c^2 is positive and c is real, and that if $\bar{\rho}'$ is positive throughout, c is purely imaginary and the fluid is unstable. Clearly the same conclusion would have been reached if equation (11) for three-dimensional disturbances had been used instead of equation (15). If $\bar{\rho}'$ is partly negative and partly positive, equation (20) still demands that c^2 be real. It must also be negative, since it must be unstable on physical grounds. If $\bar{\rho}$ is discontinuous, the terms

$$-\frac{gd}{c^2}\sum_{i=1}^{M} (\Delta\bar{\rho})_i f_i^2$$

* That this limit exists can be proved by the aid of an easily constructed density distribution, which by Sturm's comparison theorem (for continuous $\bar{\rho}$) must have an eigenvalue (finite) greater than those for the sequence.

† The simplest way to see this is to note that k may be assigned negative values, so that whatever the sign of the imaginary part of c, the waves will grow.

must be added to the left-hand side of equation (20), and the conclusions are unchanged. In the above expression, i indicates the ith interface, $\Delta\bar{\rho}$ is the density jump (negative), and M is the total number of density discontinuities. The terms to be added arise from integration by parts and application of equation (18). Alternatively and preferably, the last integral in (20) can be considered to be in the Stieltjes sense. In this discussion, the density of the top layer can be taken to be zero without introducing any difficulty.

For an inversion of an originally stable stratification or, equivalently, for a reversal of the direction of gravity, equation (15) shows that the eigenvalue c^2 must change sign but retains its magnitude. Thus the fluid will be unstable, and the amplification factor $-ikc$ is exactly the same as the time frequency $\sigma\,(=kc)$ for the stable stratification. Calculations of phase velocities for given k's therefore also provides results for the amplification factor when the stratification is unstable. In the following sections, $\bar{\rho}'$ will be assumed to be negative throughout.

Without any detailed calculations, conclusions can be at once drawn from equations (15) and (16) that for the same stratification and the same mode* the smaller the wave-number m (hence the longer the wave-length), the greater the phase velocity c. This conclusion is the direct consequence of Sturm's fundamental theorem (Ince 1944, pp. 224–5) that the solutions of

$$\frac{d}{dx}\left(K\frac{df}{dx}\right)-Gf=0 \tag{21}$$

oscillate more rapidly when K and G are diminished algebraically. In the present discussion K is $\bar{\rho}$, which is the same for all wave-lengths. The quantity G is

$$m^2\bar{\rho}+\frac{gd}{c^2}\bar{\rho}',$$

which certainly diminishes as m diminishes. Inspection of the proof of the theorem reveals that it is still valid for the specified K and G if $\bar{\rho}$ has finite discontinuities. For the greatest value of c, the boundary conditions call for exactly one oscillation in the interval (0, 1). If for a certain m and c the boundary conditions are satisfied, for a smaller m and the same gd and c there would be one zero at $\eta = 0$ (as required) and one other between 0 and 1, but not at 1, since the new G would be uniformly smaller than the old. In order to satisfy the boundary condition at $\eta = 1$, c must be greater if gd is the same, since G increases uniformly with c. The fastest waves are therefore the longest waves. If the fluid is flowing at a uniform velocity U greater than the phase velocity of the longest waves of gravest mode, no infinitesimal disturbances can travel upstream. The flow is then supercritical and internal hydraulic jumps may occur under suitable downstream conditions. The reason for this is that finite disturbances travel faster than infinitesimal ones, and can travel with a speed equal to U to make a stationary jump possible. If U is less than the greatest possible c but larger than the greatest c for higher modes, internal hydraulic jumps may still occur.

From equations (15) and (16) it can be seen immediately that for a given $\bar{\rho}(\eta)$ and a given m, c is simply proportional to $\sqrt{(gd)}$. The values of gd/c^2 are deter-

* The mode of the wave is determined by the number of zeros of the function f.

mined by the governing differential system, and are the eigenvalues, the lowest of which corresponds to the greatest possible phase velocity. According to Sturm's main theorem of oscillation (Ince 1944, p. 233), the number of zeros in the open interval $0 < \eta < 1$ for the function $f(\eta)$ is greater by one as the index (arranged according to increasing magnitude) of the eigenvalue for $\lambda = gd/c^2$ is increased by one. (Since the end-points are always zeros by specification, the number of zeros in the closed interval $0 \leqslant \eta \leqslant 1$ also increases by one when the index of λ is increased by one.) If the first mode is associated with the first eigenvalue, etc., higher and higher modes correspond to more and more nodal planes of wave motion, and smaller and smaller phase velocities. In the following sections, the definition of 'mode' given above will be retained throughout, even in the presence of surfaces of density discontinuity. Whenever *distinguishable*, waves principally associated with surfaces of discontinuity are called interfacial waves in general or free-surface waves in particular (when the density on one side is very small), and waves principally associated with continuous stratifications are called internal waves.

Sturm's fundamental theorem is again useful for comparing the phase velocities for the same wave-number but different stratifications. According to the theorem, for the same m and gd/c^2, the number of zeros of $f(\eta)$ for a smaller $\bar{\rho}$ and a greater $|\bar{\rho}'|$ in the interval (0, 1) is at least as great as that for a greater $\bar{\rho}$ and smaller $|\bar{\rho}'|$. This means that if the eigenvalues for $\bar{\rho}_1$ and $\bar{\rho}_2$ are λ_1 and λ_2, respectively, and if $\bar{\rho}_1 > \bar{\rho}_2$ and $|\bar{\rho}_1'| < |\bar{\rho}_2'|$, then $\lambda_2 < \lambda_1$ and $c_2 > c_1$. This is very understandable from a physical point of view. From a review of the derivation of equation (15), it is clear that $\bar{\rho}$ is associated with the role of density as a measure of inertia, whereas $g\bar{\rho}'$ is a measure of the restoring force responsible for the existence of wave motion. A smaller $\bar{\rho}$ and greater $|\bar{\rho}'|$ therefore correspond to a greater time frequency of oscillation and (for the same m) a greater phase velocity. If density discontinuities are present, the comparison theorem is useful only if they occur at the same locations for the two stratifications under comparison. In that case the inequality for $\bar{\rho}'$ must be supplemented by

$$|\Delta\bar{\rho}|_1 < |\Delta\bar{\rho}|_2$$

at all locations of density discontinuities.

With the aid of the Sturm–Liouville theory, the ranges in which phase velocities for the various modes must lie can be determined for the case of continuous density. If the lowest density is a and the highest density b, and if the algebraically least and greatest values of the density gradient are $-\beta$ and $-\beta+\epsilon$, then for the nth mode (with $n+1$ zeros in the closed interval $0 \leqslant \eta \leqslant 1$)

$$\frac{gd\beta}{ac^2} - m^2 \geqslant n^2\pi^2, \quad \frac{gd(\beta-\epsilon)}{bc^2} - m^2 \leqslant (n+1)^2\pi^2,$$

or

$$\frac{gd\beta}{a(n^2\pi^2+m^2)} \geqslant c^2 \geqslant \frac{gd(\beta-\epsilon)}{b[(n+1)^2\pi^2+m^2]}. \tag{22}$$

Although these inequalities have been derived for continuous $\bar{\rho}$, they are still valid in the presence of density jumps, provided the $n+1$ zeros all occur in one layer with continuous density. For small ϵ and small $b-a$, and for large n or m, the inequalities give a rather sharp estimate of c. These inequalities are obtained by comparing the zeros of the eigenfunctions with those of sine functions (which are solutions of a Sturm–Liouville system), and by applying Sturm's fundamental theorem (Ince 1944, p. 227.)

4. Surfaces of density discontinuity

It has been observed (Lamb 1945) that for two superposed layers of homogeneous fluids differing slightly in density and with a free surface on top, there are two distinct modes of gravity waves. For the one mode the phase velocity and the amplitude distribution with height are nearly the same as those of waves propagating on the free surface of a homogeneous fluid, the slight density difference of the two fluids producing only a slight correction. For the other mode the situation is entirely different. The free surface is now nearly horizontal, with negligible waviness, the greatest amplitude occurs at the interface, and the phase velocity is very much smaller. These conclusions follow from detailed calculations given in Lamb's book. What will happen in the general case of many interfaces (not excluding a free surface) separating many continuously but slightly stratified layers? Is it possible to reach similar conclusions? And, if so, is it possible to do so without detailed calculations? The answers to these questions are in the affirmative.

The simplest case of two continuously and slightly stratified layers with a single interface (which can be considered a free-surface if $\bar{\rho}_u$ is small) will be considered first. The first eigenvalue for $\lambda\ (= gd/c^2)$ calls for exactly two zeros situated at the end-points. Thus, the first mode corresponds essentially to interfacial waves only slightly affected by the slight continuous density variations. The subsequent modes are markedly different from the first one. For the next mode there is a zero of $f(\eta)$ between 0 and 1 (Sturm's oscillation theorem; see Ince (1944), p. 233). Since the continuous density gradient is small throughout, the inequalities (22) immediately show that $c^2 \sim \beta$ and is very small. Since f' is of the order* of 1, equation (18) shows that f is very small at the interface and of the order c^2. Equation (17) then shows that ζ_0 is of the order of c, and is therefore small. Thus the interface is almost horizontal, as if it were a rigid surface. For subsequent eigenvalues the number of zeros continually increases, and the same argument can be applied to reach the same conclusion. Furthermore, from the inequalities (22) it can be seen that, even if the density gradient is not very small, the phase velocity is still small (and hence the interface still behaves essentially as a rigid boundary) at large wave-numbers for any mode, and for high modes at the same wave-number.

The case next in complexity is that of three continuously stratified layers contained between two rigid boundaries and separated by two surfaces of density discontinuity. This includes the case of two layers with a free surface

* The function f can of course be multiplied by any constant. For convenience, its maximum value will be taken to be of order 1. For not too high a mode $f'(\eta)$ will then be of order 1. If the mode is high, f' may be considerably greater than 1, but then β is supposed to be very small, and the higher the mode, the smaller c^2 is for the same β. In fact, the product c^2f'/gd is of the order of β/n.

on top, for the highest of the three layers can be considered to consist of a fluid of very small density. For the first mode, the only zeros of $f(\eta)$ occur at $\eta = 0$ and $\eta = 1$. By Sturm's oscillation theorem another zero must appear in the interval (0, 1) for the second eigenvalue. In which layer will this zero appear? Depending on the thicknesses of the layers and the magnitude of the density gradient in each layer, this middle zero can occur in any of the three layers. However, for very small density gradients in the layers, it can be concluded that the middle zero must occur in the middle layer. For, with the density discontinuities at the interfaces kept constant, the density gradient in each layer can be made to approach zero. As these gradients become smaller and smaller, a stage will be reached when the middle zero will be situated in the middle layer and will stay there as the gradients are further reduced, since otherwise either the highest or the lowest layer would have two zeros, and, in the limiting case, with two zeros in a homogeneous fluid, there could be no wave motion. Only one mode of wave motion would then exist in the limiting case of three homogeneous layers of fluids, in contradiction to established facts.

If the continuous density gradients are not small, liberal sufficient conditions can be found under which the middle zero must be located in the middle layer. In the case of three layers of thicknesses d_1, d_2 and d_3, it can be shown that

$$\left[\frac{(-\bar{\rho}')_{\min}}{\bar{\rho}_{\max}}\right]_{\text{upper layers}} > \left(\frac{d_3}{d_1+d_2}\right)^2 \left[\frac{(-\bar{\rho}')_{\max}}{\bar{\rho}_{\min}}\right]_{\text{lowest layer}} \tag{23}$$

and

$$\left[\frac{(-\bar{\rho}')_{\min}}{\bar{\rho}_{\max}}\right]_{\text{lower layers}} > \left(\frac{d_1}{d_2+d_3}\right)^2 \left[\frac{(-\bar{\rho}')_{\max}}{\bar{\rho}_{\min}}\right]_{\text{top layer}} \tag{24}$$

are sufficient conditions for long waves. If there is indeed an additional zero in the lowest layer, then Sturm's oscillation theorem states that

$$\left[\frac{g(-\bar{\rho}')_{\max}}{c^2\bar{\rho}_{\min}}\right]_{\text{lowest layer}} > \frac{\pi^2}{d_3^2}. \tag{25}$$

From equation (23) it then follows that

$$\left[\frac{g(-\bar{\rho}')_{\min}}{c^2\bar{\rho}_{\max}}\right]_{\text{upper layers}} > \frac{\pi^2}{(d_1+d_2)^2}, \tag{26}$$

which is sufficient to guarantee an additional zero (other than the one at $\eta = 1$) in the upper two layers, in contradiction to hypothesis. Hence the middle zero must not occur in the bottom layer under the stated condition. The proof for the statement concerning equation (24) is entirely similar. For the general case of n layers of thicknesses $d_i\,(i = 1, \ldots, n)$, criteria similar to inequalities (25) and (26) can be used to determine the regions in which the additional zero or zeros must be located.

For the case of three layers, the next higher mode brings in a fourth zero. If the previous three zeros are in different layers, this fourth one must cause one of the layers to have two zeros. Then the inequalities (22) show that for small density gradients (small β) c^2 must indeed be very small. From the interfacial conditions stated by equation (18) it can be seen that the two interfaces now behave almost like rigid boundaries—a situation which is maintained in the subsequent modes. For the general case, it can be shown by reasoning similar to that employed in the preceding two paragraphs that, for very small density variation in each layer, zeros in addition to those at the end-points will appear one after the other as the modes are higher and higher, until each layer has one and only one zero. The next higher eigenvalue, bringing in another zero, corresponds to a mode with one of its layers having two nodal points (or zeros). The surfaces of density discontinuity will then behave like rigid boundaries, and will so behave for subsequent modes. The number of modes for which the surfaces of density discontinuity do not behave like rigid boundaries is $n-1$ for n layers of fluid—the same as the number of these surfaces. The phase velocities for these modes are, for small density variation in each layer, nearly the same as those for wave motion in n homogeneous fluid layers with the same depths and the same mean densities as the layers under consideration.

The conclusions reached in this section remain essentially valid if the effect of surface tension Γ is included in the interfacial boundary conditions, which then assume the form

$$(\bar{\rho}f')_u-(\bar{\rho}f')_l+\left[\frac{\Gamma m^2}{c^2 d}+(\bar{\rho}_l-\bar{\rho}_u)\frac{gd}{c^2}\right]f=0. \tag{27}$$

If there is only one surface of density discontinuity and if the density gradient elsewhere is small, the effect of surface tension, as far as interfacial waves are concerned, is to increase g by the amount

$$\frac{\Gamma m^2}{(\bar{\rho}_l-\bar{\rho}_u)\,d^2}=\frac{\Gamma k^2}{\bar{\rho}_l-\bar{\rho}_u}.$$

If there are two or more surfaces of discontinuity, approximate phase velocities of the interfacial waves can be found by assuming the density in each layer to be constant and equal to the mean density (provided that the density variation in each layer is small), and by imposing the boundary conditions (27) instead of (18). For internal waves the effect of surface tension is to stiffen the interfaces further (as can also be seen from the fact that the coefficient of f in equation (27) is increased by surface tension), and if the interfaces behave like rigid barriers without surface tension, they will do so even more effectively with the aid of surface tension. Thus the phase velocities for truly internal waves are hardly affected by surface tension.

5. Equipartition of energy

Equation (20), used in a previous section to prove the reality of c, also contains the theorem of equipartition of energy. For isopycnic particles

$$(u,w,\zeta)=\left[-f'(z),\,ikf(z),\,-\frac{f(z)}{c}\right]\exp ik(x-ct).$$

Integration of u^2, w^2 and ζ^2 with respect to x over a wave-length produces the common factor $\frac{1}{2}$. With the multiplication of this common factor, equation (20) can be written, in dimensional terms

$$\frac{g}{2}\int_0^d \bar{\rho}'\zeta^2\,dz=\frac{1}{2}\left(\int_0^d \bar{\rho}f'^2\,dz+k^2\int_0^d \bar{\rho}f^2\,dz\right). \tag{28}$$

The left-hand side of this equation represents the excess of potential energy* per unit wave-length in the x-direction over that of the mean configuration, or the potential energy of wave motion. The right-hand side obviously represents the kinetic energy per unit length. The equation therefore states that there is equipartition of energy. Although the proof given is for progressive waves, a slight modification of the form of u, w and ζ will make it valid for standing waves also.

Equation (20) was derived for the case of rigid barriers at $z = 0$ and $z = d$, and for a continuous stratification. If there are density discontinuities, the integral on the left-hand side of equation (28) must be taken to be in the Stieltjes sense, but the equipartition of energy is unaffected. In this discussion, the density of the top layer can be taken to be zero without introducing any difficulty.

For the same wave-number m general wave motion in a stratified fluid may be composed of many modes, each of which corresponds to an eigenvalue of gd/c^2 (or of c). It will be shown that the kinetic and potential energies of the component modes are entirely separable. In other words there are no energy couplings between the component modes at all. The proof for the case of a continuously stratified fluid between rigid boundaries is straightforward. The differential system consisting of equations (15) and (16) is self-adjoint, and it is well known that for such a system the eigenfunctions are orthogonal. In the present instance this means that (with r and s indicating the modes)

$$\int_0^1 \bar{\rho}' f_r(\eta) f_s(\eta)\, d\eta = 0, \tag{29}$$

or that there is no coupling of the various modes as far as potential energy of the wave motion is concerned. To obtain the corresponding result for kinetic energy, equation (15) will be written in the form

$$(\bar{\rho} f_r')' - (m^2 \bar{\rho} + \lambda_r \bar{\rho}') f_r = 0. \tag{30}$$

If equation (30) is multiplied by f_s and integrated, and if equations (16) are utilized, the result

$$\int_0^1 \bar{\rho} (f_r' f_s' + m^2 f_r f_s)\, d\eta = 0 \tag{31}$$

is obtained, which states that there is also no coupling between the different modes in connexion with kinetic energy.

In the presence of surfaces of density discontinuity, the governing differential system is no longer self-adjoint in the ordinary sense, but the de-coupling of potential and kinetic energies can be proved with the aid of the boundary conditions (18). Cross-multiplication of equation (30) and the equation

$$(\bar{\rho} f_s')' - (m^2 \bar{\rho} + \lambda_s \bar{\rho}') f_s = 0 \tag{32}$$

* The factor $\frac{1}{2}$ can best be explained by considering the potential energy of water waves. The water in the troughs are raised to the crests, each element ζdx by the height ζ. Thus the total potential energy is proportional to the integral of ζ^2 over a half wave-length. Hence the factor $\frac{1}{2}$.

by f_s and f_r and integration of the difference of the results produces, after utilization of equation (20) at all the interfaces,

$$\sum_{i=1}^{M} (\Delta\bar{\rho} f_r f_s)_i + \sum_{j=1}^{n} \int_{\eta_{j-1}}^{\eta_j} \bar{\rho}' f_r f_s d\eta = 0, \tag{33}$$

which is the same as equation (29) if the latter is considered to be in the Stieltjes sense. From equations (30) and (33), equation (31) is again obtained. Thus the normality of the energy spectra is established even in the presence of density discontinuities, as indeed is to be expected.

So far the normality of the energy spectra has been established only for the same wave-number. It remains to mention that if the wave-numbers are different, the net coupling effect is finite over a distance in the x-direction however long, and is therefore zero per unit distance.

The conclusion of equipartition of energy is valid for three-dimensional waves also. The proof is entirely similar. One needs only to start with equation (11) instead of equation (15), and use $lu = kv$.

6. The case of infinite depth

As has been remarked by Lamb (1945) for incompressible fluids, and as can be easily verified from equation (9a) for a semi-infinite fluid with any stratification (whether compressible or not), the system consisting of equation (9a) (with $l = 0$ and w changed to f), the conditions at infinity

$$f(z) \to 0 \quad \text{as} \quad z \to -\infty, \tag{34}$$

and the *usual* free-surface condition

$$f'(0) - \frac{g}{c^2} f(0) = 0 \tag{35}$$

possess the solution
$$f(z) = e^{kz}, \quad c^2 = g/k. \tag{36}$$

The fact that the solution is independent of the mean density distribution and that it corresponds to irrotational motion is certainly rather surprising. But the explanation is not far to seek. The solution is the same as the well-known solution for wave motion in a semi-infinite homogeneous fluid. If it is valid also for a non-homogeneous fluid, there must be something very special about it. The special feature is that in co-ordinates *moving with the waves* the streamlines of the (steady) flow corresponding to the solution are lines of constant pressure, as can be readily demonstrated. This situation is not affected if the density is a function of the stream function alone (which is the case for an incompressible fluid in steady motion) or for a compressible fluid with isentropic change of state along a streamline (hence p is a function of ρ alone along a streamline). The distribution of mean density or mean entropy in the vertical direction is quite immaterial, so long as the density or entropy does not change along a streamline in the moving frame of reference. For convenience only two-dimensional waves have been discussed. The essential features of the flow are, however, unchanged if the waves are three-dimensional.

Actually, though the solution given by equation (36) is the only one which is independent of density stratification, another interesting one exists if c_s is constant. Equation (13), with w changed to f, and the boundary conditions (34) and (35) are evidently satisfied by

$$f(z) = \exp\{(g/c_s^2)\,z\}, \quad c = c_s. \tag{37}$$

The motion represented by this solution, which is incidentally independent of the wave-number, is not irrotational. That it is independent of the density variation (so long as it is consistent with constant c_s or with an isothermal atmosphere) is again because streamlines in a frame moving with the waves are isobars. It has often been said that sound waves are longitudinal. The wave motion discovered here propagates with sound velocity, and is strongly affected by compressibility, and yet, since c_s is large and therefore $u \sim f'(z)$ is small, it is predominately transverse.

In the case of finite depth the eigenvalues are discrete and, if the stratification is continuous over any portion of the fluid, infinite in number. If the depth is infinite, the spectrum of the eigenvalues is continuous, with a number of discrete eigenvalues equal to the number of density discontinuities (or possibly greater than it if the fluid is compressible). For demonstration, a fluid with the density variation

$$\bar{\rho} = \rho_0 e^{-bz} \quad (b \text{ positive})$$

and with a free surface at $z = 0$ may be considered. For this particular density variation, the atmosphere is isothermal and the sound velocity constant, as can be seen by applying the hydrostatic equation $dp = -g\bar{\rho}\,dz$. Thus equation (13) can be solved exactly. (The free surface should, of course, be removed to $z = \infty$ to make the problem realistic.) However, the purpose of this section is to show the continuity of the spectrum of σ. Hence the fluid will be assumed to be incompressible for simplicity, and the fluid to extend to $z = -\infty$. Equation (12a) possesses the solution

$$f(z) = A\,e^{a_1 z} + B^{a_2 z}$$

in which

$$(a_1, a_2) = \frac{1}{2}\left[b \pm \sqrt{\left\{b^2 - 4k^2\left(\frac{gb}{\sigma^2} - 1\right)\right\}}\right] \quad (\sigma = kc). \tag{38}$$

If $gb < \sigma^2$, equation (34) demands that $B = 0$, and the surface condition (35) demands that

$$a_1 = \frac{gk^2}{\sigma^2} \quad \text{or} \quad c^2 = \frac{g}{k}.$$

The solution is therefore the one discussed before in the present section. However, if $gb > \sigma^2$, both a_1 and a_2 have a positive real part, so that the condition at $z = -\infty$ is automatically satisfied, and the surface condition assumes the form

$$Aa_1 + Ba_2 - (A+B)\frac{gk^2}{\sigma^2} = 0.$$

Given any k, b and σ, B can be solved in terms of A or vice versa. Thus any σ equal to or less than $\sqrt{(gb)}$ will do, and the spectrum of σ is continuous from zero to $\sqrt{(gb)}$. For any k the spectrum of c is then continuous from zero to $\sqrt{(gb)}/k$. For $k < b$, the value gk for σ^2 satisfies the requirement $gb/\sigma^2 \geqslant 1$. Thus the situation can be summarized as follows. (1) For any k, gk is an eigenvalue for σ^2. It corresponds to a motion identical to that of ordinary surface waves, and is an isolated mode if $k > b$. (2) Any σ^2 less than gb corresponds to a possible wave motion. For $k \leqslant b$, the eigenvalue (for σ^2) for the free-surface mode is imbedded in the continuous spectrum for σ^2.

The example just given is instructive in showing that free-surface waves are not necessarily faster than internal waves. In fact, even if the depth is finite, free-surface waves or interfacial waves (corresponding to discontinuous density changes) are not necessarily faster than internal waves that owe their existence to continuous stratifications. They are faster only if the gradients of the continuous density stratifications are small.

If the top of the fluid is covered with a rigid boundary, no solution is possible for $gb \leqslant \sigma^2$. The continuous spectrum is given by

$$\frac{gb}{\sigma^2} > 1, \quad \text{or} \quad \sigma^2 < gb.$$

These values of σ^2 all correspond to internal waves. The free-surface mode is removed with the removal of the free surface, as expected.

7. Calculation of phase velocities

The phase velocities of waves propagating in a stratified fluid can be calculated rapidly by the method of Ritz. The differential system determining $\lambda = gd/c^2$ for a given density variation and a given wave-number consists of equations (15) and (16) for the case of fixed boundaries. Equation (15) is a special case of the general equation

$$L(f) = \lambda G(\eta) f, \tag{39}$$

in which L is a linear operator. According to the Ritz method, a set of N linearly independent functions $\phi_r(\eta)$ satisfying the boundary conditions will be chosen. The quantities

$$A_{rs} = -\int_0^1 \phi_r(\eta)\, L[\phi_s(\eta)]\, d\eta \tag{40}$$

and

$$B_{rs} = -\int_0^1 G(\eta)\, \phi_r(\eta)\, \phi_s(\eta)\, d\eta \tag{41}$$

are symmetric in the sense that their values are unaffected as r and s are interchanged. This is obvious with B_{rs}. With A_{rs} symmetry is immediately clear upon integration by parts and utilization of the boundary conditions. The integrals in equations (40) and (41) are in the Stieltjes sense when applied to wave propagation in a stratified fluid with density jumps. (For a similar application, see Courant & Hilbert (1931, Vol. I. pp. 349–50). In fact, in case a free surface exists, it is not necessary to assume that there is a layer of light fluid with non-zero density over it. The choice of $\phi(\eta)$ can therefore be less restricted, see Courant (1926).) The eigenfunction is now approximated by a linear combination of the chosen functions

$$F(\eta) = \sum_{r=1}^{N} c_r \phi_r(\eta). \tag{42}$$

With f in equation (39) replaced by F, that equation is multiplied by F and integrated to yield

$$\int_0^1 FL(F)\, d\eta + \mu \int_0^1 \bar{\rho}' F^2\, d\eta = \sum_{r,s=1}^{N} (A_{rs} - \mu B_{rs})\, c_r c_s.$$

The right-hand side of this equation depends only on the c's. If the condition is imposed that it be stationary in value for variations in the c's, the N equations are obtained

$$\sum_{r=1}^{N} (A_{rs} - \mu B_{rs}) c_r = 0, \quad (s = 1, 2, \ldots, N).$$

Since the c's must not all vanish, it is necessary that

$$\begin{vmatrix} A_{11} - \mu B_{11} & A_{12} - \mu B_{12} & \ldots & A_{1N} - \mu B_{1N} \\ A_{21} - \mu B_{21} & A_{22} - \mu B_{22} & \ldots & A_{2N} - \mu B_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ A_{N1} - \mu B_{N1} & A_{N2} - \mu B_{N2} & \ldots & A_{NN} - \mu B_{NN} \end{vmatrix} = 0. \tag{43}$$

Ritz has proved that the N roots (for μ) of this equation are near and slightly larger than the first N eigenvalues of λ. In general, if $N + M$ terms are taken for F in equation (42) the first N roots of the enlarged determinantal equation are nearer and nearer the first N eigenvalues as M becomes larger and larger. Thus, if only N terms are taken, the accuracy of the roots as eigenvalues increases in general from the Nth to the first root.

Two examples (for continuous density distribution) will be given to demonstrate the accuracy of the Ritz method. For the first, the density variation is assumed to be

$$\bar{\rho} = \rho_0 e^{-\beta\eta}.$$

Equation (15) can then be reduced to the form

$$f'' - \beta f' - \left(m^2 - \frac{\beta g d}{c^2}\right) f = 0. \tag{44}$$

With

$$(\alpha_1, \alpha_2) = \frac{1}{2}\left[\beta \pm \sqrt{\left\{\beta^2 - 4\left(\frac{\beta g d}{c^2} - m^2\right)\right\}}\right],$$

the solution satisfying the boundary condition at $\eta = 0$ is

$$f = A(e^{\alpha_1 \eta} - e^{\alpha_2 \eta}).$$

The condition $f(1) = 0$ then demands that

$$e^{\alpha_1} - e^{\alpha_2} = 0,$$

or

$$\alpha_1 - \alpha_2 = \sqrt{\{\beta^2 - 4(\beta\lambda - m^2)\}} = 2n\pi i \quad (n = 1, 2, \ldots).$$

Hence

$$4\beta\lambda = 4n^2\pi^2 + \beta^2 + 4m^2,$$

or, with

$$\lambda' = \beta\lambda,$$

$$\lambda' = n^2\pi^2 + m^2 + \tfrac{1}{4}\beta^2. \tag{45}$$

To use the Ritz method, it is better to write equation (44) in its original form, before the exponential factor was cancelled out

$$(e^{-\beta\eta} f')' - m^2 e^{-\beta\eta} f = -\lambda e^{-\beta\eta} f.$$

If $\phi_r(\eta) = \sin r\pi\eta$ and the first element of the determinant in equation (43) is equated to zero, the result (after cancelling a common factor) is obtained

$$2\pi^4 + m^2(2\pi^2 - \beta^2) = \mu_1(2\pi^2 - \beta^2),$$

or, for small β,

$$\mu_1 = \pi^2 + m^2 + \tfrac{1}{2}\beta^2.$$

This is greater than the first eigenvalue only by the amount $\frac{1}{4}\beta^2$. That roots of equation (43) are always greater than the first eigenvalue is well known, but it is important to remember that this is so only if the differential equation is written in the form of equation (15), i.e. in the self-adjoint form. Had equation (44) been used for the application of the Ritz method, the value π^2+m^2 would have been obtained for μ_1, which is *smaller* than the first eigenvalue.

Had the true eigenfunctions $e^{\frac{1}{2}\beta\eta}\sin r\pi\eta$ been chosen to be $\phi_r(\eta)$, the true eigenvalues would of course be obtained. However, the purpose of this example is to demonstrate the power of Ritz's method in the general case, in which the form of the eigenfunctions is not known and cannot be easily guessed.

A second example is provided by the density variation

$$\bar{\rho}(\eta) = \rho_0\sqrt{(1-\beta\eta)}.$$

An exact solution exists in this case for long waves. Equation (15) can be written, for zero wave-number,

$$[\sqrt{(1-\beta\eta)}f']' + \frac{\beta\lambda}{2\sqrt{(1-\beta\eta)}}f = 0,$$

or

$$\frac{d^2f}{d\xi^2} + \frac{2\lambda}{\beta}\{1-\sqrt{(1-\beta)}\}^2 f = 0,$$

with

$$\xi = \frac{1-\sqrt{(1-\beta\eta)}}{1-\sqrt{(1-\beta)}}.$$

The boundary conditions are

$$f(\xi) = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = 1.$$

The eigenvalues for long waves are exactly

$$\lambda' = \beta\lambda = \frac{1}{2}\left[\frac{\beta n\pi}{1-\sqrt{(1-\beta)}}\right]^2.$$

Here again, if $\sin r\pi\xi$ were chosen to be $\phi_r(\eta)$ in the application of Ritz's method, the true eigenvalues would be obtained. This will not be done, for the same reason as stated before. Instead, we choose

$$\phi_r(\eta) = (1-\beta\eta)^{\frac{1}{4}}\sin r\pi\eta.$$

Then

$$B_{11} = \tfrac{1}{4},$$

$$A_{11} = \int_0^1 \left[\pi^2(1-\beta\eta)\cos^2\pi\eta - \frac{\beta\pi}{2}\sin\pi\eta\cos\pi\eta + \frac{\beta^2}{16}\frac{\sin^2\pi\eta}{1-\beta\eta}\right]d\eta$$

$$= \frac{\pi^2}{2} - \frac{\beta\pi^2}{4} + \frac{\beta^2}{32}\int_0^1 \frac{1-\cos 2\pi\eta}{1-\beta\eta}\,d\eta$$

$$= \frac{\pi^2}{2} - \frac{\beta\pi^2}{4} - \frac{\beta}{32}\ln(1-\beta) + O(\beta^4).$$

Therefore

$$\mu_1 = 2\pi^2 - \beta\pi^2 + \frac{\beta^2}{8} - \frac{\beta^3}{16} + O(\beta^4),$$

whereas

$$\lambda_1' = 2\pi^2 - \beta\pi^2 - \frac{\beta^2\pi^2}{8} + \frac{3\pi^2}{32}\beta^3 + O(\beta^4),$$

showing that the error is again of the order of β^2. The two examples given demonstrate that the modified Froude number $c/\sqrt{(g'd)}$ (with g' equal to $(g/\bar{\rho})\,d\bar{\rho}/d\eta$) for a small and practically constant density gradient is approximately equal to π^{-1} for long waves—a fact of great importance for prediction of the existence of lee waves in the wake of a body advancing in a stratified fluid. The number π^{-1}, however, cannot be applied to fluids whose density gradient is not practically constant.

It has been shown that for n continuously stratified layers with $n-1$ or n surfaces of density discontinuity there are $n-1$ or n modes of motion corresponding to interfacial (or free-surface) waves. If the density gradient in each layer is small, approximations to the first $n-1$ or n eigenvalues corresponding to these modes can be found by Greenhill's formula on the assumption that the density in each layer is constant and equal to the mean of the actual density in that layer. Since it has been shown that for higher modes the surfaces of density discontinuity behave like solid barriers, the subsequent eigenvalues can be found without appreciable error by considering each layer to be bounded by solid planes, and by arranging the eigenvalues λ found separately for all the layers in ascending order of magnitude.

To test the validity of this statement, one may consider the case of two layers of depth d_1 and d_2, and with density variations given by

$$\bar{\rho}_1(\eta_1) = C_1 e^{-\beta_1\eta_1}, \quad \bar{\rho}_2(\eta_2) = C_2 e^{-\beta_2\eta_2},$$

in which

$$\eta_1 = \frac{z}{d_1}, \quad \eta_2 = \frac{z}{d_2},$$

the subscript 1 is for the upper layer, and the origin of z is taken at the interface. The differential equation governing the motion in each layer is equation (44), and the solutions satisfying the boundary conditions $f_1(1) = 0$ and $f_2(-1) = 0$ are

$$f_1(\eta_1) = A(e^{\alpha_1\eta_1} - e^{(\alpha_2-\alpha_1)+\alpha_2\eta}),$$

$$f_2(\eta_2) = B(e^{\gamma_1\eta_2} - e^{(\gamma_2-\gamma_1)+\gamma_2\eta}),$$

with

$$(\alpha_1, \alpha_2) = \frac{1}{2}\left[\beta_1 \pm \sqrt{\left\{\beta_1^2 - 4\left(\frac{\beta_1 g d_1}{c^2} - m^2\right)\right\}}\right],$$

$$(\gamma_1, \gamma_2) = \frac{1}{2}\left[\beta_2 \pm \sqrt{\left\{\beta_2^2 - 4\left(\frac{\beta_2 g d}{c^2} - m^2\right)\right\}}\right].$$

Imposing the interfacial conditions

$$f_1(0) = f_2(0) \quad \text{and} \quad \frac{C_1}{d_1}f_1'(0) - \frac{C_2}{d_2}f_2'(0) + (C_2 - C_1)\frac{g}{c^2}f_1(0) = 0,$$

one has

$$\frac{C_1}{d_1}(\alpha_1 - \alpha_2 e^{\alpha_2-\alpha_1}) - \frac{C_2}{d_2}\frac{1-e^{\alpha_2-\alpha_1}}{1-e^{\gamma_2-\gamma_1}}(\gamma_1 - \gamma_2 e^{\gamma_2-\gamma_1}) + (C_2 - C_1)\frac{g}{c^2}(1 - e^{\alpha_2-\alpha_1}) = 0, \quad (46a)$$

or, alternatively,

$$\frac{C_1}{d_1}\frac{1-e^{\gamma_2-\gamma_1}}{1-e^{\alpha_2-\alpha_1}}(\alpha_1 - \alpha_2 e^{\alpha_2-\alpha_1}) - \frac{C_2}{d_2}(\gamma_1 - \gamma_2 e^{\gamma_2-\gamma_1}) + (C_2 - C_1)\frac{g}{c^2}(1 - e^{\gamma_2-\gamma_1}) = 0. \quad (46b)$$

The question is now asked: what error will be committed if g/c^2 is calculated from equation (45) for each layer, on the assumption that the interface is rigid? For the upper layer

$$\frac{g}{c^2} = \frac{1}{\beta_1 d_1}(n^2\pi^2 + m^2 + \tfrac{1}{4}\beta_1^2) \tag{45a}$$

according to equation (45), which is large* for small β_1. The dominant term in equation (46) is therefore the last term, and one sequence of phase velocities is given approximately by

$$1 - e^{\alpha_2 - \alpha_1} = 0,$$

which is the condition that would be obtained for the upper layer if the interface were rigid. The error committed is the first term, and is of the magnitude $2n\pi C_1/d_1$ which can be annihilated by changing c^2 slightly (and therefore also making $1 - e^{\alpha_2-\alpha_1}$ slightly different from zero). One may, in fact, multiply equation (46*a*) by $c^2/(C_2 - C_1)g$ and note that the error committed in calculating c^2/g by (45*a*) is of the order of $[2n\pi C_1/(C_2 - C_1)][\beta_1/(n^2\pi^2 + m^2)]$, which is small if β_1 is small or if n or m is large. For the lower layer, equation (46*b*) can be used for proving that little error is committed if equation (45) is used to calculate c, provided β_2 is small or n or m large. Thus the entire phase-velocity spectrum of internal waves (i.e. with the interfacial one excepted) is obtained by superimposing the phase-velocity spectrum of one layer on that of the other, both obtained on the supposition that the interfaces were rigid. Although the example is a specific one, the conclusion reached is evidently valid in general, in virtue of the results obtained in §4.

If, however, the density difference in each layer is not small compared with the mean density, phase velocities can only be obtained by solving the entire eigenvalue problem. This can be done either approximately, by the use of the Ritz method, or analytically, by solving the differential equation for the bottom layer with the restriction that $f(0) = 0$ and with $f'(0)$ arbitrary. In the analytical solution, when the first interface is reached the starting f for the next layer is made to be the same as the terminal f for the bottom layer, and the starting f' for the new layer is found from the first interfacial boundary condition. This procedure is continued until the upper boundary, free or rigid, is reached. The final boundary condition is imposed and the eigenvalues for λ are found from the final equation obtained.

8. Waves generated by a plane wave-maker

The motion generated in a stratified fluid contained between two rigid boundaries by an oscillating plane normal to these boundaries and extending all the way between them can be found by solving

$$(\bar{\rho}f')' - m^2\left(\bar{\rho} + \frac{g}{\sigma^2 d}\bar{\rho}'\right)f = 0, \tag{15a}$$

with σ now equal to 2π times the oscillation frequency of the wave-maker. The eigenvalues are now those of m^2 consistent with equations (16). If σ is sufficiently large, the quantity in the second parenthesis of equation (15*a*) is positive, and the

* Since interval waves are being discussed, c^2 is very small even according to the exact equations (46), as can be asserted by virtue of the inequalities (22). Therefore the statement that g/c^2 is large for small β_1 is not based on a circular argument.

eigenvalues will be negative. These correspond to imaginary m's and exponential (instead of sinusoidal) dependence of the disturbance on x. If σ is very small, the quantity referred to above is negative, and the eigenvalues for m^2 are positive. The m will be real and waves will propagate from the oscillating plane. For the intermediate case of medium σ, the quantity in the second parenthesis in equation (15a) is positive for certain levels, and negative at others. In this case the eigenvalues of m^2 go from negative infinity (through discrete values) to positive infinity.

If the oscillating plane is situated at $x = 0$ and oscillates as $a\cos\sigma t$, and the fluid extends from there to $x = \infty$, the solution for the motion in general consists of a parallel-flow part, a part corresponding to the local disturbance, and a part corresponding to wave motion. The parallel-flow part is necessary in order to satisfy the demand of continuity and is given by

$$-a^2\sigma\left(\int_0^1 \frac{d\eta}{\bar{\rho}}\right)^{-1}\int_0^\eta \frac{d\eta}{\bar{\rho}},$$

which obviously satisfies equation (15a), with m equal to zero (not an eigenvalue). The solution for the stream function is then, with Sommerfield's radiation condition at infinity,

$$\frac{\psi}{a^2\sigma} = -\cos\sigma t\left(\int_0^1 \frac{d\eta}{\bar{\rho}}\right)^{-1}\int_0^\eta \frac{d\eta}{\bar{\rho}} + \sum_{n=-\infty}^{n_1-1} A_n f_n(\eta)\, e^{-|m_n|x}\cos\sigma t + \sum_{n=n_1}^{\infty} A_n f_n(\eta)\cos(m_n x - \sigma t), \tag{47}$$

in which n_1 is the index of the first positive eigenvalue for m^2, and the coefficients A are determined by the condition at the oscillating plane

$$-\eta = -\left(\int_0^1 \frac{d\eta}{\bar{\rho}}\right)^{-1}\int_0^\eta \frac{d\eta}{\bar{\rho}} + \sum_{n=-\infty}^{\infty} A_n f_n(\eta),$$

by the usual method, since the eigenfunctions f_n are orthogonal. Since the f's are eigenfunctions of a Sturm–Liouville system, and since the latter are known to be complete, the completeness of the f's is not in question. A similar solution can be obtained if density jumps are present, and if the wave-maker has any shape and any specified motion whatever.

The mean energy flux at infinity can be calculated either directly or by means of the group velocity for each wave component. (The formula for calculating group velocity from phase velocity is the usual one.) The mean rate of work done by the wave-maker is equal to this mean energy flux. If all the m's are imaginary, there is no wave motion and no mean energy flux at infinity. Hence the mean rate of work done by the oscillating plane is zero.

The sign of the second parenthesis, which decides whether the eigenvalues of m^2 are positive or negative, or partly positive and partly negative, is directly connected with the type of the partial differential equation governing the motion. For incompressible fluids in two-dimensional motion with a time dependence described by $e^{-i\sigma t}$, equation (8) assumes the form

$$(\sigma^2\bar{\rho} + g\bar{\rho}')\frac{\partial^2 w}{\partial x^2} + \sigma^2\frac{\partial^2}{\partial z^2}\left(\bar{\rho}\frac{\partial w}{\partial z}\right) = 0.$$

If this is multiplied by $\bar{\rho}$ and the new variable

$$\xi = \int_0^z \frac{dz}{\bar{\rho}}$$

is used, the equation

$$\left(\sigma^2\bar{\rho}^2 + g\bar{\rho}\frac{d\bar{\rho}}{dz}\right)\frac{\partial^2 w}{\partial x^2} + \sigma^2\frac{\partial^2 w}{\partial \xi^2} = 0 \tag{48}$$

is obtained, which is elliptic or hyperbolic according as (Görtler 1954)

$$\sigma^2\bar{\rho} + g\frac{d\bar{\rho}}{dz}$$

is positive or negative. But this quantity is proportional to

$$\bar{\rho} + \frac{g}{\sigma^2 d}\frac{d\bar{\rho}}{d\eta}.$$

Consequently the sign of the second parenthesis in equation (15*a*) determines the type of equation (48). However, the type of the partial differential equation governing the motion of a stratified fluid, though of course relevant to the type of solution obtained (as shown in this section), does not play as significant a role as the type of the partial differential equation governing the homentropic flow of a compressible fluid. The reason is simply that for a given mode of wave motion corresponding to a $[\bar{\rho} + (g/\sigma^2 d)(d\bar{\rho}/d\eta)]$ with alternating signs, the particle velocities being small, the velocity in the steady flow relative to the waves is simply c everywhere, and the elliptic and hyperbolic regions do not in any sense correspond to subsonic and supersonic flows.

When equation (48) is hyperbolic, real characteristics exist, with slopes given by

$$\tan\gamma' = \frac{\sigma}{[-g\bar{\rho}(d\bar{\rho}/dz) - \sigma^2\bar{\rho}^2]^{\frac{1}{2}}}.$$

For very small σ, the characteristics are horizontal. This is in agreement with the finding (Yih 1959*a*) that steady, two-dimensional flows of an inviscid stratified fluid become one-dimensional when the motion is weak—a phenomenon that has been experimentally demonstrated (Yih 1959*b*).

9. Stability of stratified liquid under vertical vibration

The stability of a homogeneous liquid with a free surface when the container is accelerated periodically has been considered by Benjamin & Ursell (1954). The cause of instability is a kind of resonance, though not in the usual sense of a forced harmonic oscillation, and the frequency of free oscillation plays a role in the determination of stability or instability. For a stratified liquid the frequencies of free oscillation are infinite in number, and it can be expected that the resonance phenomenon has to be investigated for each mode of free oscillation, i.e. for each of the eigen-frequencies in the absence of the imposed vibration. That this is indeed the case can be seen from the following analysis.

The equations of motion *relative to the vibrating container* are

$$\bar{\rho}\frac{\partial}{\partial t}(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p - \rho(0, 0, g - F\cos\omega t), \tag{49}$$

in which $F\cos\omega t$ is the acceleration of the container. Cross-differentiation of the first two of equations (49) again produces an equation which shows that $\bar{\rho}u$ and $\bar{\rho}v$ possess a potential ϕ, so that

$$\bar{\rho}(u,v) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\phi.$$

By the same procedure as that employed in § 2, an equation similar to equation (6) is obtained

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z}+\bar{\rho}w\right)+(g-F\cos\omega t)\rho = 0. \tag{50}$$

The equation of continuity now has the form

$$\bar{\rho}\frac{\partial w}{\partial z} = \nabla^2\phi,$$

and the equation of incompressibility is

$$\frac{\partial\rho}{\partial t} = -w\bar{\rho}',$$

with the prime indicating differentiation with respect to z. These equations are applicable even to gases if the effect of compressibility is small. Applying the Laplacian operator in x and y to equation (50), dividing throughout by $g-F\cos\omega t$, differentiating the result with respect to t, and utilizing the equations of continuity and of incompressibility, one has

$$\frac{\partial}{\partial t}\frac{\dfrac{\partial}{\partial t}\left[\dfrac{\partial}{\partial z}\left(\bar{\rho}\dfrac{\partial w}{\partial z}\right)+\bar{\rho}\nabla^2 w\right]}{g-F\cos\omega t} = \bar{\rho}'\nabla^2 w. \tag{51}$$

With equation (7) and

$$w = A(t)\,S(x,y)\,w(z),$$

equation (51) can be written, with w denoting $w(z)$ for simplicity,

$$\frac{\partial}{\partial t}\frac{(\partial/\partial t)\,A[(\bar{\rho}w')'-\alpha^2\bar{\rho}w]}{g-F\cos\omega t} = -\alpha^2\bar{\rho}'wA. \tag{52}$$

This shows that

$$(\bar{\rho}w')'-\alpha^2\bar{\rho}w = C\bar{\rho}'w \tag{53}$$

and

$$\frac{d}{dt}\left(\frac{dA/dt}{g-F\cos\omega t}\right) = -\frac{\alpha^2}{C}A. \tag{54}$$

For a continuously stratified fluid bounded by two horizontal planes of the container (at distance d apart), C must be $g\alpha^2/\sigma^2$ in which σ^2 is exactly the eigenvalue in equation (15a), which is simply the dimensionless form (with w changed to f and for the special case of two-dimensional flow) of

$$(\bar{\rho}w')'-\alpha^2\left(\bar{\rho}+\frac{g\bar{\rho}'}{\sigma^2}\right)w = 0, \tag{55}$$

for only when C assumes such a value can the boundary conditions $w(0) = w(d) = 0$ be satisfied. Since surfaces of density discontinuity can be considered as limiting cases of regions of large density gradients, C must be equal to

$g\alpha^2/\sigma^2$ even in the presence of density discontinuities, so long as σ^2 is understood to be the eigenvalue of (55) for the given stratification, and so long as the effect of surface tension is neglected. Consequently, in the absence of surface tension, equation (54) has the form

$$\frac{d}{dt}\left(\frac{dA/dt}{g-F\cos\omega t}\right) = -\frac{\sigma^2}{g}A. \tag{56}$$

A more convenient equation than equation (56) for investigation of stability is the equation for the amplitude function $a(t)$ of the deflexion ζ of any material particle from its mean position. Since $w = \partial\zeta/\partial t$, we have $A = da/dt$, and integration of equation (56) yields

$$\frac{d^2a}{dt^2} = -\frac{\sigma^2}{g}(g-F\cos\omega t)\,a+C'. \tag{57}$$

Initially, u, v, w, p, ρ and ζ are all zero, so that a is zero and, from equation (49),

$$\frac{\partial w}{\partial t} = \frac{\partial^2\zeta}{\partial t^2} = 0$$

or

$$\frac{d^2a}{dt^2} = 0.$$

Consequently $C' = 0$, and, with $T = \frac{1}{2}\omega t$, equation (57) becomes

$$\frac{d^2a}{dT^2}+(p-2q\cos 2T)\,a = 0, \tag{58}$$

in which (p not indicating pressure)

$$p = \frac{4\sigma^2}{\omega^2}, \quad q = \frac{2\sigma^2F}{\omega^2g}. \tag{59}$$

Equation (58) is Mathieu's equation in its standard form. If q vanishes, equation (58) shows that the frequency of oscillation is $\sigma/2\pi$, as expected. The quantity F/ω^2 is the linear amplitude of the vibration of the container. Hence $q = (2\sigma^2/g)\times$ (amplitude of vibration) (compare with Benjamin & Ursell 1954).

Whether the fluid is stable or not depends on whether $a(T)$ remains bounded as $T\to\infty$, and this in turn depends on p and q. The regions of stability and instability are shown in figure 1. Only the first quadrant of the diagram is shown because both p and q are positive. In fact, the complete diagram in McLachlan's book (1947, p. 41) shows that the diagram is symmetric about the p-axes. The unshaded regions are stable regions and the shaded ones unstable regions. Apart from an exponential factor $e^{\mu T}$ (μ depending on p and q) indicating the rate of growth, the solution for the unstable cases also possesses exact periodicities (see McLachlan 1947, pp. 40, 41, 77, 78). In the lowest shaded part of figure 1, the period for T is 2π, so that the period for ωt is 4π. This means that the frequency of fluid oscillation is only half the frequency of the container. In the next shaded region, the period for T is π, so that the oscillation of the fluid and that of the container are isochronous. The third shaded region is a region of half-frequency, etc. The stable regions would be regions of half-frequency or isochronous regions but for a factor $e^{i\beta T}$, with β dependent on p and q. If β is irrational, the solution is not periodic at all.

Since
$$\frac{p}{q} = \frac{2g}{F},$$
the points in the p–q plane to be considered in each case are all on a straight line radiating from the origin. For *each** α there are infinitely many σ, and therefore infinitely many points on that straight line. Whereas some points may lie in regions of instability, others may lie in regions of stability. Since σ decreases

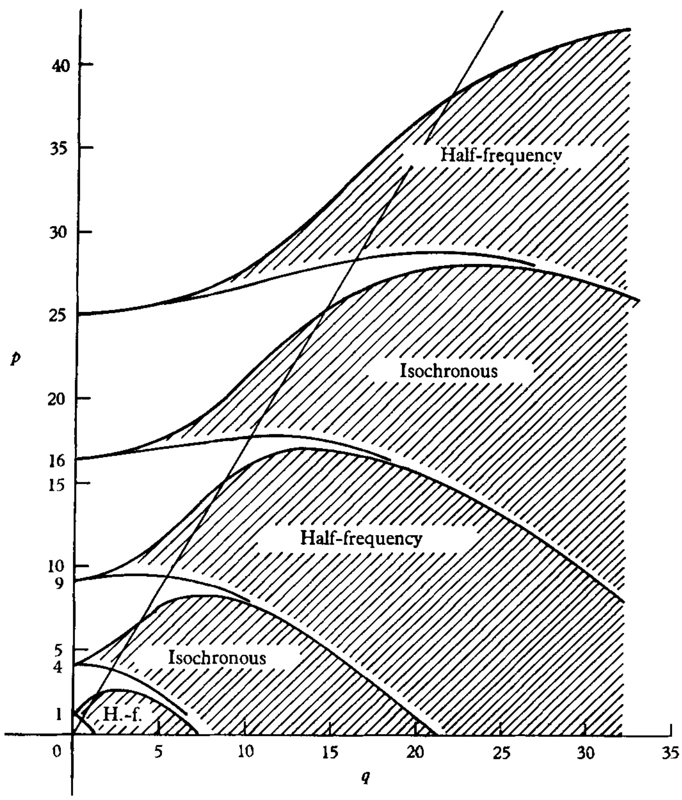

FIGURE 1. Chart for stability of fluid under vertical vibration. The acceleration of the container is $F \cos \omega t$, σ is the frequency (radians per second) of free oscillation, and $p = 4\sigma^2/\omega^2$, $q = 2\sigma^2 F/\omega^2 g$. The points for different modes lie on the line $p/q = 2g/F$.

toward zero as the index of the mode is higher and higher, the origin is a limit point of the (infinitely many) points in the p–q plane whose locations determine stability or instability. From figure 1 it is clear that there is a small region of stability around the origin, so that the fluid is stable for sufficiently high driving frequency, and against resonance with sufficiently high modes of free oscillation. Significant is the fact that no matter how small F is, there is always a region of instability, though the region is smaller and smaller as F becomes smaller and smaller.

* In the case treated by Benjamin & Ursell for each α there is only one σ.

So far surface tension has been omitted from the discussion. If the amount of density discontinuity ($\Delta\rho$) is constant, and if the fluid in each layer is homogeneous, the effect of surface tension can be taken into account in a very simple manner. The differential equation governing the flow in each layer is the Laplace equation, satisfied by the potential function $\phi(x, y, z)$. The boundary conditions at the interfaces are (compare with equation (20a))

$$\left(\rho\frac{\partial\phi}{\partial t}\right)_l - \left(\rho\frac{\partial\phi}{\partial t}\right)_u - (g - F\cos\omega t)\,\Delta\rho\zeta + \Gamma\nabla^2\zeta = 0. \tag{60}$$

Since $\nabla^2\zeta = -\alpha^2\zeta$, this is enough to show that in the absence of continuous stratification, and if $\Delta\rho$ is constant, the effect of Γ is to increase g by the amount $\alpha^2\Gamma/\Delta\rho$. Thus, under the stated restrictions, the stability of superposed layers of homogeneous fluids can be studied by changing g to $g + (\alpha^2\Gamma/\Delta\rho)$, and subsequently applying the results obtained for the case of no surface tension. In particular, if there is only one layer of fluid with a free surface (with $\Delta\rho = \rho$)

$$\sigma^2 = \alpha \tanh \alpha d\left(g + \frac{\alpha^2\Gamma}{\rho}\right),$$

so that
$$p = \frac{4\alpha \tanh \alpha d}{\omega^2}\left(g + \frac{\alpha^2\Gamma}{\rho}\right), \quad q = \frac{2\alpha F \tanh \alpha d}{\omega^2},$$

in agreement with the results of Benjamin & Ursell. The case of two homogeneous fluids bounded by two rigid barriers and having an interface can be treated similarly. If the thickness of each layer is $\frac{1}{2}d$

$$\sigma^2 = \frac{\Delta\rho}{\rho_1 + \rho_2}\alpha \tanh\frac{\alpha d}{2}\left(g + \frac{\alpha^2\Gamma}{\Delta\rho}\right),$$

and p and q are again given by equation (59).

If there is continuous density change as well as discontinuous ones, or if each layer is homogeneous but the density discontinuities ($\Delta\rho$) are not constant, the effect of surface tension becomes rather difficult to determine—especially in the former case. However, it seems unlikely, in view of the behaviour of surfaces of density discontinuity when the fluid is undergoing free oscillation with *small* frequencies, that surface tension will have an appreciable effect on the 'resonance' of the imposed acceleration with these free oscillations. In other words, for very small σ's, it is reasonable to expect that the stability or instability can be decided by ignoring surface tension entirely. Additional research in this direction is necessary before more definite and more general conclusions can be drawn as to the effect of surface tension on the kind of stability under discussion.

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